

Equivalence of Minkowski and Euclidean Field Theory Solutions

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Abstract

We consider the correspondence between solutions of non-gravitational field theories formulated in Euclidean space-time and Minkowski space-time. Infinitely many “Euclidean” spaces can be obtained from M^4 via a group of transformations in which the Wick rotation is a special case. We then discuss how the solutions of gauge field theories formulated in these “Euclidean” spaces have a one-to-one correspondence with the solutions of field theories formulated in Minkowski space-time, provided we avoid the one-point compactification into $S^4 = E^4 \cup \infty$.

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To solve technical problems encountered in phenomenological and theoretical applications of non-gravitational field theories formulated in flat Minkowski space-time with Lorentz metric $(+ - - -)$ it is often useful to transform to the four-dimensional Euclidean space-time with metric $(+ + + +)$. Solutions are then found in the Euclidean space-time. We discuss the problem of determining whether or not these solutions have corresponding solutions in Minkowski space-time.

Normally, Euclidean space-time is reached via the Wick rotation, which is a special case of the more general transformation given by $\tau = \alpha_1 t + \alpha_2$, where $\alpha_1, \alpha_2 \in \mathbb{C}$. Each transformation is an element of the group of conformal transformations of the plane onto itself. To obtain the so-called generalized Wick rotation, we take $\alpha_1 = e^{i\pi/2}$ and $\alpha_2 = 0$.

Our intention is to demonstrate that field theory solutions in the Euclidean space-time obtained above have a one-to-one correspondence with solutions in Minkowski space-time provided that the conditions imposed at infinity in the Euclidean space-time do not amount to a one-point compactification into $S^4 = E^4 \cup \infty$. Let the gauge theory in a Euclidean or Minkowski space-time manifold be described by a principal bundle (B, P, π, F, G) . B is the *base space*, which is E^4 for theories formulated in Euclidean space-time and M^4 for theories formulated in Minkowski space-time. P is the *total space* over B and F is a *typical fiber* of P . The projection which maps a fiber F from P to a point in B is π . G is the *gauge group*, which is always topologically equivalent to the typical fiber F in a principal bundle. The fiber over a point p in the Euclidean base space contains solutions in every possible gauge at point p . A single point on a fiber represents a solution in a particular gauge. Different gauges can be reached by operating on a point in the fiber by an element in gauge group G . For a curve in B the solutions of the theory in a chosen gauge will be represented by single points on each of the fibers above the curve. The locus of these points is a cross-section.

Consider the principal fiber bundle $(M^4, P_M, \pi_M, F_M, G_M)$ describing a Minkowski field theory. Because there is a homeomorphic mapping f (Wick rotation) between Euclidean space-time E^4 and the Minkowski base space M^4 , a unique mapping from P_M to a new space P_E exists. The induced pullback bundle P_E forms a total space over E^4 with the same typical fiber as P_M . The total spaces P_E and P_M are topologically equivalent and the cross-sections are homeomorphic.[1] Now, the cross-section of the pullback bundle P_E is the pullback of the cross-section of P_M . Therefore the field

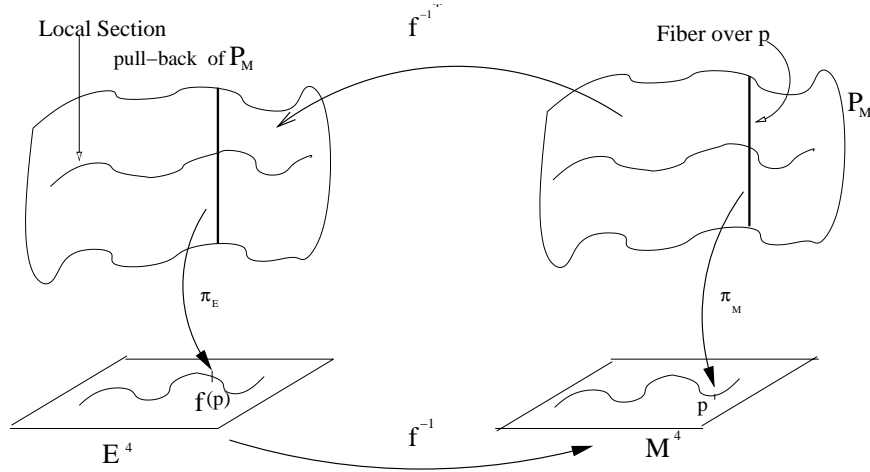


Figure 1: A Wick rotation f maps M^4 to E^4 . A field theory in M^4 is described by *total space* P_M . The corresponding field theory in E^4 is contained in a pullback bundle, which carries the same *typical fiber* as P_M . Therefore the two principal bundles have the same gauge group. The solutions for each *base space* in a chosen gauge form a section.

theory solutions have a one-to-one correspondence. We also note that the base spaces E^4 and M^4 are contractable spaces so the two principal fiber bundles formed are *trivial* and hence global cross-sections exist.

In some formal and phenomenological applications of Euclidean field theories, assumptions have to be made about the asymptotic behavior of the field. In order to achieve finite action for theories written in E^4 , appropriate conditions are imposed at infinity (i.e. $r \equiv \sqrt{\sum_{i=1}^4 x_i^2} \rightarrow \infty$). If these conditions are equivalent to the one-point compactification $S^4 = E^4 \cup \infty$, then there no longer exists a homeomorphism between the base spaces, i.e. there is no mapping between S^4 and M^4 unless a pole is removed from S^4 . Therefore the total space over S^4 is not the pullback of P_M . S^4 is not contractable and the total space over S^4 is not *trivial*. Furthermore, Singer[2] showed that the principal bundle with base space S^4 and a compact, nonabelian Lie group as structure group admits no global sections. A gauge cannot be chosen that gives a continuous section everywhere in the total space above S^4 . This problem is known as the Gribov Ambiguity[3]. Since P_M admits global sections, there is no homeomorphism between P_M and the total space

over S^4 . This implies that there is no one-to-one correspondence between the field theory solutions of M^4 and S^4 . Subsequent compactification of M^4 does not preserve the correspondence since a compactification of M^4 yields a topologically different space $S^1 \times S^3$ with a flat Lorentz metric[4]. This lack of correspondence is seen in nonabelian theories over $E^4 \cup \infty$. Such theories find self-dual and anti-self-dual solutions satisfying $F = \pm *F$, where F is the curvature two-form and $*F$ is the dual. In M^4 , where this relation becomes $*F = \pm iF$, no solution corresponding to a compact Lie group can be found[5].

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